SASTA OLIPHANT
SCIENCE AWARDS

Prize Winner

Programming, Apps & Robotics

Year 9-10

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| Simulating Three Bodies Interacting Through Gravity

Vinuka Kaluwila – Programming Apps and Robotics

| Section 1: Introduction

There is one force in our lives that we experience every day, but hardly ever stop to think about – gravity. From shaping the motions of galaxies over millennia to keeping us all on the ground each day, gravity fundamentally influences the universe we live in. But how exactly *does* gravity affect the movements of objects in space?

| Section 2: Aim and Scientific Purpose

The aim of this project is to be used as a teaching resource, to educate students about how gravity is simulated, equations in 3 dimensions, and more broadly about how differential equations are used to simulate real-world scenarios.

| Section 3: Potential Uses

This simulation is not intended to be used practically for simulating the motions of planets and stars, but rather as an education tool to show *how* these motions are calculated on a larger scale when scientists and researchers professionally calculate orbits and planetary motion. Hence, this program can potentially be used in schools or universities to educate students about *how* gravity is simulated and the complexities of the underlying mathematics and science.

| Section 4: Requirements to Run the Program

This program is made on Desmos online graphing calculator, which is compatible with almost all online browsers, including the latest two versions of Chrome, Firefox, Safari, and Microsoft Edge. Similarly, internet access is required to run the program as Demos is an online tool. For best results, the simulation can be run on a laptop.

| Section 5: Instructions for Use

1. Open the program on: https://www.desmos.com/calculator/etzdz3idyc
2. Scroll down using the scroll bar on the left side of the simulation, until the folder "Constants (Mass, G, and Δt)" is reached. Click on the grey triangle near the folder icon to open the folder.
3. Set these constants to the desired values, using the sliders insider the folders. m_1 is the mass of the red body, m_2 is the mass of the green body, and m_3 is the mass of the blue body. Increasing the mass will increase the gravitational attraction between bodies. G is the gravitational constant, with higher values of G corresponding to more gravitational attraction between bodies. In real conditions, $G \approx 6.6743 \times 10^{-11}$. While this will make the simulation reflective of actual conditions in space, the simulation will run extremely slowly (as gravity is, in actuality, an incredibly weak force). Hence, $G = 1$ is the value which is set by default in the simulation. When these constants are set to desired values, this folder can be closed by pressing the triangle icon again.

4. Scroll down once again using the scroll bar on the left side of the simulation, until the folders "Position 01," "Velocity 01," "Position 02," "Velocity 02," "Position 03," and "Velocity 03" are reached.
5. Open each of the folders one by one and use the sliders to change the different initial positions, and velocities! The initial positions of the red body, the green body, and the blue body are $M_1(x_1, y_1, z_1)$, $M_2(x_2, y_2, z_2)$ and $M_3(x_3, y_3, z_3)$ respectively. The values of $x_1, y_1, z_1 \ldots y_3, z_3$ can be changed to desired values, using the sliders inside the folders. The velocity vectors for the red body, green body, and blue body are stored as V_1, V_2 and V_3, with initial conditions $V_1 = [v_{1x}, v_{1y}, v_{1z}]$, $V_2 = [v_{2x}, v_{2y}, v_{2z}]$ and $V_3 = [v_{3x}, v_{3y}, v_{3z}]$. These values of $v_{1x}, v_{1y}, v_{1z} \ldots v_{3y}, v_{3z}$ can also all be changed using the sliders inside the folders. When these initial conditions are set to desired values, the folders can be closed by pressing the triangle icons again.
6. Press the "Reset" button on the right of the simulation. **This is very important as this sets the initial conditions. You should see the positions and velocity vectors change on the right side of the simulation once this is pressed.**
7. Run the simulation by pressing the "Start/Stop" button and drag the white point inside the square on the right of the simulation to rotate the $3D$ space as the simulation is running.
8. When you want to stop the simulation, press the "Start/Stop" button again, and to restore the initial conditions the "Reset button" can be pressed again.
9. Folders can be turned off by pressing the folder icon. This can be used to turn off the axes by pressing the "Visualizing the 3D space" folder, or can be used to turn off the velocity vectors by pressing the "Velocity Vectors" folder. Similarly, there is information inside the folders on how the simulation is run, and how the differential equations are numerically solved!
10. If there are any questions, there are instructions on the program itself, inside the folders. Similarly, feel free to contact me on vinuka.kaluwila@pembroke.sa.edu.au if there are any problems or queries.

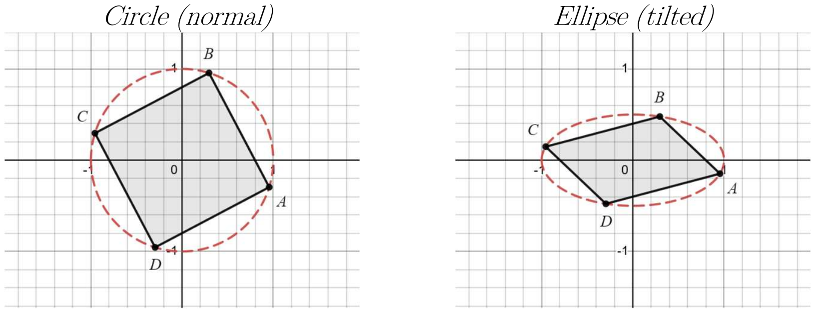
| Section 6: Explanation of Code and Underlying Mathematics/Science

| Section 6.1.1: Creating the xy plane

When making a simulation of three bodies interacting in three dimensions, the first step was to create a three-dimensional graphing system within the Desmos graphing calculator (which originally graphs in two dimensions). This was be done by projecting points in three dimensions to the two-dimensional grid.

As can be seen in Figure 01, a rotatable square can be made by connecting four points around a circle which are all $\frac{\pi}{2}$ radians (90°) of a rotation apart from each other. Furthermore, tilting that square along a horizontal axis simply has the effect of morphing the circle into an ellipse.

Figure 01: Showing tilting a square result in its bounding circle being morphed into an ellipse.



To achieve the first freedom of rotation, it was noted that points on a circle are of the form $(\cos(\phi), \sin(\phi))$, the four points A, B, C and D of a square, can be written as $A(\cos(\phi), \sin(\phi))$, $B(\cos(\phi + \frac{\pi}{2}), \sin(\phi + \frac{\pi}{2}))$, $C(\cos(\phi + \pi), \sin(\phi + \pi))$ and $D(\cos(\phi + \frac{3\pi}{2}), \sin(\phi + \frac{3\pi}{2}))$, as they must all be $\frac{\pi}{2}$ radians apart from each other. Using the fact that both $\sin(x)$ and $\cos(x)$ are simply $\frac{\pi}{2}$ radians out of phase from each other, this can be simplified to give $A(\cos(\phi), \sin(\phi))$, $B(-\sin(\phi), \cos(\phi))$, $C(-\cos(\phi), -\sin(\phi))$ and $D(\sin(\phi), -\cos(\phi))$. Hence, by varying ϕ between $-\frac{\pi}{4}$ and $\frac{7\pi}{4}$ the square could be made rotatable.

To achieve the tilt, it was noted that the equation for an ellipse with major radius of 1 is of the form $x^2 + \frac{y^2}{k^2} = 1$, where k relates to how much the ellipse is "squished" and hence how much the square appears tilted into the higher dimension. As k only affects how much the y-coordinates of A, B, C and D are "squished," the y-coordinates of these points must be multiplied by a scale factor of k, where k ranges between -1 and 1.

This results in the following points for A, B, C and D, where ϕ is the rotation of the square ranging between $-\frac{\pi}{4}$ and $\frac{7\pi}{4}$, and k is the tilt of the square into the third dimension between -1 and 1:

$$A(\cos(\phi), k\sin(\phi))$$

$$B(-\sin(\phi), k\cos(\phi))$$

$$C(-\cos(\phi), -k\sin(\phi))$$

$$D(\sin(\phi), -k\cos(\phi))$$

These points were connected using Desmos' "polygon" tool, which allowed for the xy plane to be drawn:

$$\operatorname{polygon}(A, B, C, D)$$

This forms the basis for the xy plane in the final simulation, as it allows for two axes of rotation given by the parameters ϕ and k. For simplicity purposes, the values for the x and y-coordinates for points A, B, C and D were given the names of A_x, A_y, B_x, B_y, C_x, C_y and D_x, D_y respectively, such that $A(A_x, A_y)$, $B(B_x, B_y)$, and so on for C and D.

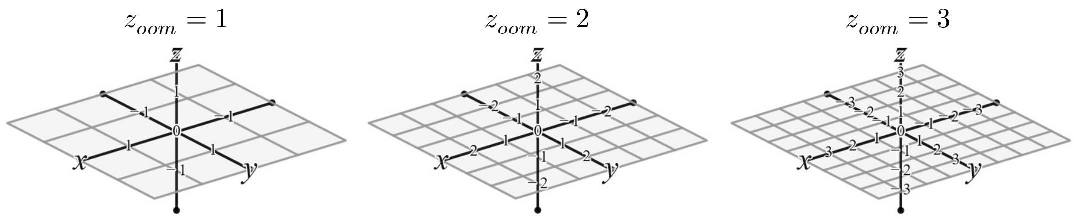
Hence, this was implemented into the simulation through the following LaTeX code, and can be found under the "Visualizing the 3D space" folder in the simulation:

\operatorname{polygon}\left(A,B,C,D\right)
A=\left(A_{x},A_{y}\right)
B=\left(B_{x},B_{y}\right)
C=\left(C_{x},C_{y}\right)
D=\left(D_{x},D_{y}\right)
A_{x}=\cos\left(\phi\right)
A_{y}=k\sin\left(\phi\right)
B_{x}=-\sin\left(\phi\right)
B_{y}=k\cos\left(\phi\right)
C_{x}=-\cos\left(\phi\right)
C_{y}=-k\sin\left(\phi\right)
D_{x}=\sin\left(\phi\right)
D_{y}=-k\cos\left(\phi\right)

| Section 6.1.2: Creating the grid

The next step in creating the simulation was to create a grid on this square, which would become the xy plane in the final simulation. First, a parameter called z_{oom} was defined, which is the maximum value of the x or y-coordinates shown on the plane. For example, the left-most part of Figure 02 shows what $z_{oom} = 1$ corresponds to, the middle part shows what $z_{oom} = 2$ corresponds to, and the right-most image shows what $z_{oom} = 3$ corresponds to, in the final plane.

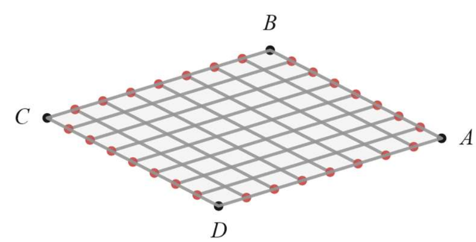
Figure 02: Showing different values of z_{oom}.



Then, a new parameter was defined called z, such that $z = 2z_{oom} + 2$. This is is analagous to the number of small squares wide the grid would be. For example, as can be seen in Figure 01, when $z_{oom} = 1$, the grid is 4 small squares wide, and when $z_{oom} = 2$, the grid is 6 small squares wide. Note that this z does not refer to the z axis.

Now, consider Figure 03. As can be seen, to create the grid on the square, the lines AB, BC, CD and AD can be split into z regions (hence $z - 1$ equally spaced points) and these points be connected to create the lines on the grid.

Figure 03: Showing how to create the lines of the grid by splitting lines AB, BC, CD and AD into z equal regions, and plotting $z - 1$ points on each line segment.



To do this, the Desmos function of "lists" were used. The list n was defined such that $n = [1,2,3 \dots z - 1]$. Now, the section formula in coordinate geometery was used. To split the line connectiong points (x_1, y_1) and (x_2, y_2) into segments of the ratios $p: q$, the resulting point is given by:

$$\left(\frac{px_2 + qx_1}{p + q}, \frac{py_2 + qy_1}{p + q}\right)$$

Hence, this gives the following expressions for the points along the lines AB, BC, CD and AD, using the list $n = [1,2,3 \dots z - 1]$:

$$\left(\frac{nA_x + (z - n)B_x}{z}, \frac{nA_y + (z - n)B_y}{z}\right)$$

$$\left(\frac{nD_x + (z - n)C_x}{z}, \frac{nD_y + (z - n)C_y}{z}\right)$$

$$\left(\frac{nA_x + (z-n)D_x}{z}, \frac{nA_y + (z-n)D_y}{z}\right)$$

$$\left(\frac{nB_x + (z-n)C_x}{z}, \frac{nB_y + (z-n)C_y}{z}\right)$$

Hence, to connect the points opposite from each other to create the lines in the grid, Desmos' "polygon" function was used. This would simply connect the corresponding points to create the lines in the grid:

$$\text{polygon}\left(\left(\frac{nA_x + (z-n)B_x}{z}, \frac{nA_y + (z-n)B_y}{z}\right), \left(\frac{nD_x + (z-n)C_x}{z}, \frac{nD_y + (z-n)C_y}{z}\right)\right)$$

$$\text{polygon}\left(\left(\frac{nA_x + (z-n)D_x}{z}, \frac{nA_y + (z-n)D_y}{z}\right), \left(\frac{nB_x + (z-n)C_x}{z}, \frac{nB_y + (z-n)C_y}{z}\right)\right)$$

Hence, this was implemented using the following LaTeX code, and can also be found under the "Visualizing the 3D space" folder in the simulation:

z=2z_{oom}+2
n=\left[1...z-1\right]
\operatorname{polygon}\left(\left(\frac{nA_{x}+\left(z-n\right)B_{x}}{z},\frac{nA_{y}+\left(z-n\right)B_{y}}{z}\right),\left(\frac{nD_{x}+\left(z-n\right)C_{x}}{z},\frac{nD_{y}+\left(z-n\right)C_{y}}{z}\right)\right)
\operatorname{polygon}\left(\left(\frac{nA_{x}+\left(z-n\right)D_{x}}{z},\frac{nA_{y}+\left(z-n\right)D_{y}}{z}\right),\left(\frac{nB_{x}+\left(z-n\right)C_{x}}{z},\frac{nB_{y}+\left(z-n\right)C_{y}}{z}\right)\right)

| Section 1.1.3: Creating and labelling the points on the x and y axes.

For the next section, Demos' "midpoint" function was used widely, hence the following notation was implemented, with X and Y represent two points:

$$M(X, Y) = \text{midpoint}(X, Y)$$

To create the x and y axes, it was noted that the points on these axes would be the midpoints of the points used to create the lines on the grid. Hence, Desmos' "midpoint" function was used. Once again, the list n was used. This resulted in the following, which would be the points on the x and y axes:

$$M\left(\left(\frac{nA_x + (z-n)B_x}{z}, \frac{nA_y + (z-n)B_y}{z}\right), \left(\frac{nD_x + (z-n)C_x}{z}, \frac{nD_y + (z-n)C_y}{z}\right)\right)$$

$$M\left(\left(\frac{nA_x + (z-n)D_x}{z}, \frac{nA_y + (z-n)D_y}{z}\right), \left(\frac{nB_x + (z-n)C_x}{z}, \frac{nB_y + (z-n)C_y}{z}\right)\right)$$

These points were labelled by creating new lists of the form $l = n - \frac{z}{2}$ and $o = \frac{z}{2} - n$ respectively.

To create the axes themselves, the midpoint of A and B and the midpoint of C and D were connected, and the midpoint of A and D and the midpoint of B and C were connected. Similarly, one of each of these points can be labelled x and y respectively to label the x and y axes. This gives the following, with square brackets representing line segments:

$$[M(A,B), M(C,D)] \text{ with the point } M(C,D) \text{ being labelled "}x\text{"}$$

$$[M(A,D), M(B,C)] \text{ with the point } M(A,D) \text{ being labelled "}y\text{"}$$

Hence, this was implemented into the simulation through the following LaTeX code, and can also be found under the “Visualizing the 3D space” folder in the simulation:

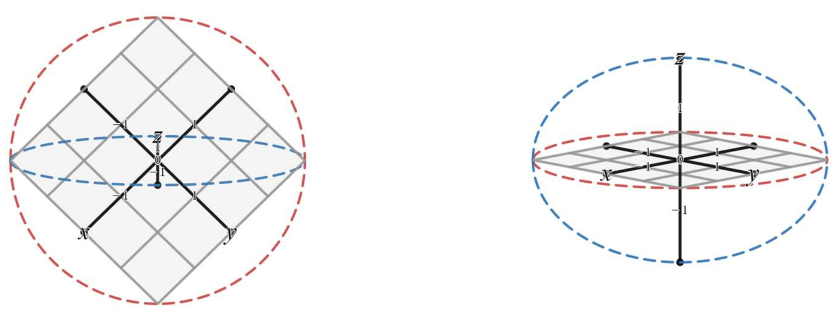
M\left(X,Y\right)=\operatorname{midpoint}\left(X,Y\right)
l=n-\frac{z}{2}
o=\frac{z}{2}-n
M\left(\left(\frac{nA_{x}+\left(z-n\right)B_{x}}{z},\frac{nA_{y}+\left(z-n\right)B_{y}}{z}\right),\left(\frac{nD_{x}+\left(z-n\right)C_{x}}{z},\frac{nD_{y}+\left(z-n\right)C_{y}}{z}\right)\right) (label: \`${l}\`)
M\left(\left(\frac{nA_{x}+\left(z-n\right)D_{x}}{z},\frac{nA_{y}+\left(z-n\right)D_{y}}{z}\right),\left(\frac{nB_{x}+\left(z-n\right)C_{x}}{z},\frac{nB_{y}+\left(z-n\right)C_{y}}{z}\right)\right) (label: \`${o}\`)
\left[M\left(A,B\right),M\left(C,D\right)\right]
M\left(C,D\right) (label: \`x\`)
\left[M\left(A,D\right),M\left(B,C\right)\right]
M\left(A,D\right) (label: \`y\`)

| Section 6.1.4: Creating the z axis

To create the z axis, it was noted that the z axis can also be modelled by an ellipse. As can be seen in Figure 04, however, when the ellipse modelling the xy plane has a low eccentricity, the z axis ellipse has a high eccentricity. Conversely, when the z axis ellipse has a lower eccentricity, the xy plane axis ellipse has a higher eccentricity. This is intuitive, as the z axis is at 90° to the x and y axes.

Figure 04: Showing the ellipse modelling the z axis (blue) along with the ellipse modelling the xy plane.

Blue ellipse has higher eccentricity *Blue ellipse has lower eccentricity*



Hence, as the ellipse modelling the xy plane depends on k^2, the ellipse modelling the z axis can be made to depend on $1-k^2$ to achieve this relationship between the two eccentricities. Hence, the ellipse modelling the z axis was made to be given by the equation $x^2 + \frac{2y^2}{1-k^2} = 1$. Note that there is a factor of 2 also affecting y^2. This is because the z axis extends both above and below the xy plane instead of just above, causing an extra coefficient of 2 to be involved. The z axis can be drawn by connecting the points at which $x = 0$, which can be solved for:

$$0^2 + \frac{2y^2}{1-k^2} = 1$$
$$2y^2 = 1 - k^2$$
$$y = \pm\frac{\sqrt{1-k^2}}{\sqrt{2}}$$

Hence, the z axis can be made by connecting the points $\left(0, \frac{\sqrt{1-k^2}}{\sqrt{2}}\right)$ and $\left(0, -\frac{\sqrt{1-k^2}}{\sqrt{2}}\right)$. This was implemented as follows:

$$\left[\left(0, \frac{\sqrt{1-k^2}}{\sqrt{2}}\right), \left(0, -\frac{\sqrt{1-k^2}}{\sqrt{2}}\right)\right] \text{ with the point } \left(0, \frac{\sqrt{1-k^2}}{\sqrt{2}}\right) \text{ being labelled “}z\text{”}$$

To number the z axis, it was noted that this could simply be done by splitting the line segment $\left[\left(0, \frac{\sqrt{1-k^2}}{\sqrt{2}}\right), \left(0, -\frac{\sqrt{1-k^2}}{\sqrt{2}}\right)\right]$ into z equal segments (not that here, z referes to the parameter defined earlier, not the z axis). This could be achieved by scaling the point $\left(0, \frac{\sqrt{1-k^2}}{\sqrt{2}}\right)$ by a factor of $\frac{2n-z}{z}$, and resulted in the following points on the z axis:

$$\left(0, \frac{(2n-z)\sqrt{1-k^2}}{\sqrt{2}z}\right)$$

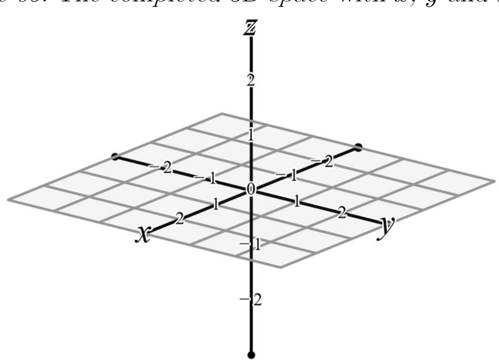
This was labelled once again using the list l defined previously.

This was implemented into the simulation through the following LaTeX code, and can also be found under the “Visualizing the 3D space” folder in the simulation:

\left[\left(0,\frac{\sqrt{1-k^{2}}}{\sqrt{2}}\right),\left(0,-\frac{\sqrt{1-k^{2}}}{\sqrt{2}}\right)\right]
\left(0,\frac{\sqrt{1-k^{2}}}{\sqrt{2}}\right) (label: \`z\`)
\left(0,\frac{\left(2n-z\right)\sqrt{1-k^{2}}}{\sqrt{2}z}\right) (label: \`${l}\`)

After completing sections 1.1.1 to 1.1.4, the result was the 3D space shown in Figure 05.

Figure 05: The completed 3D space with x, y and z axis.



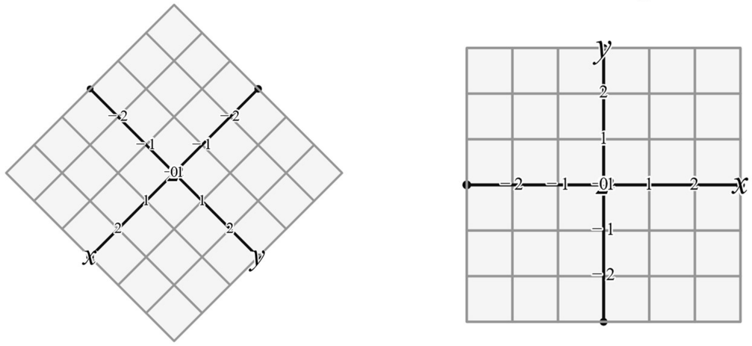
| Section 6.1.5: Creating plottable points

After projecting 3D space into 2D space, a method needed to be designed for points to be plotted on the new axes. To do this, a projection function P projecting 3D points of the form (X, Y, Z) into 2D in Desmos needed to be designed. Such a projection function would take the form $P(X, Y, Z) = (\alpha, \beta)$ taking the x, y and z-coordinates of the point as inputs and outputting a new x and y-coordinates α and β. We can construct such a function.

First, consider a point (X,Y) on the regular xy plane. This can be expressed in the polar form $(R,\theta) = (\sqrt{X^2+Y^2}, \tan^{-1}(\frac{Y}{X}))$. The simulated xy plane rotates points by $\phi - \frac{3\pi}{4}$. This is because $\phi = \frac{3\pi}{4}$ corresponds to the simulated xy plane being oriented in the same way as the normal xy plane. This is shown in Figure 06.

Figure 06: orientation of the simulated xy plane.

$\phi = 0$ $\qquad\qquad$ $\phi = \frac{3\pi}{4}$



Hence, $\phi - \frac{3\pi}{4}$ can be added to θ, to give $(R, \theta + \phi - \frac{3\pi}{4}) = (\sqrt{X^2+Y^2}, \tan^{-1}(\frac{Y}{X}) + \phi - \frac{3\pi}{4})$. Converting this back to cartesian form using $(X,Y) = (R\cos(\theta), R\sin(\theta))$ gives the basis for the projecting function. As this is not the final projection function, this was notated as P^*.

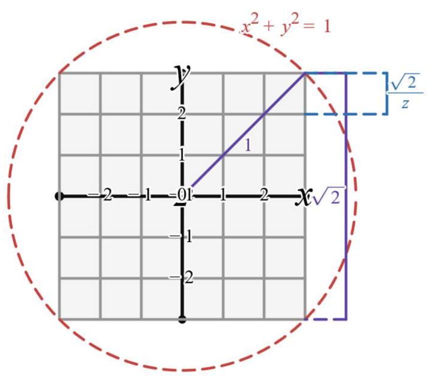
$$P^*(X,Y,0) = \left(\sqrt{X^2+Y^2}\cos\left(\tan^{-1}\left(\frac{Y}{X}\right) + \phi - \frac{3\pi}{4}\right), \sqrt{X^2+Y^2}\sin\left(\tan^{-1}\left(\frac{Y}{X}\right) + \phi - \frac{3\pi}{4}\right)\right)$$

Now, as the y-coordinate is scaled by k to give the second axis of rotation, this can be applied to the projection function. This gives:

$$P^*(X,Y,0) = \left(\sqrt{X^2+Y^2}\cos\left(\tan^{-1}\left(\frac{Y}{X}\right) + \phi - \frac{3\pi}{4}\right), k\sqrt{X^2+Y^2}\sin\left(\tan^{-1}\left(\frac{Y}{X}\right) + \phi - \frac{3\pi}{4}\right)\right)$$

Furthermore, note that the entire point needs to be scaled by $\frac{\sqrt{2}}{z}$, due to the geometry shown in Figure 07, whereby 1 increment in the simulated xy plane is equal to $\frac{\sqrt{2}}{z}$ increments in the original xy plane.

Figure 07: Geometric proof that the points need to be scaled by a factor of $\frac{\sqrt{2}}{z}$.



Apart from the scale factor of $\frac{\sqrt{2}}{z}$, it was found that when $X < 0$, the point also needed to be scaled by -1. This is due to the nature of the inverse tangent function involved. Hence, these scale factors were implemented:

$$P^*(X,Y,0)=\left(\{X\ge 0\colon 1,X<0\colon -1\}\frac{1}{z}\sqrt{2X^2+2Y^2}\cos\left(\tan^{-1}\left(\frac{Y}{X}\right)+\phi-\frac{3\pi}{4}\right),\{X\ge 0\colon 1,X<0\colon -1\}\frac{1}{z}k\sqrt{2X^2+2Y^2}\sin\left(\tan^{-1}\left(\frac{Y}{X}\right)+\phi-\frac{3\pi}{4}\right)\right)$$

Now, Z needed to be accounted for. This was relatively easy, as it was previously found that the z axis can be made by connecting the points $\left(0,\frac{\sqrt{1-k^2}}{\sqrt{2}}\right)$ and $\left(0,-\frac{\sqrt{1-k^2}}{\sqrt{2}}\right)$. Hence $\frac{2Z\sqrt{1-k^2}}{\sqrt{2}z}$ or $\frac{Z\sqrt{2-2k^2}}{z}$ can be added to the y-coordinate. This gives the final projection function, and $\frac{\sqrt{2}}{z}$ can be factored out:

$$P^*(X,Y,Z)=\left(\{X\ge 0\colon 1,X<0\colon -1\}\frac{1}{z}\sqrt{2X^2+2Y^2}\cos\left(\tan^{-1}\left(\frac{Y}{X}\right)+\phi-\frac{3\pi}{4}\right),\{X\ge 0\colon 1,X<0\colon -1\}\frac{1}{z}k\sqrt{2X^2+2Y^2}\sin\left(\tan^{-1}\left(\frac{Y}{X}\right)+\phi-\frac{3\pi}{4}\right)+\frac{Z\sqrt{2-2k^2}}{z}\right)$$

$$P^*(X,Y,Z)=\frac{\sqrt{2}}{z}\left(\{X\ge 0\colon 1,X<0\colon -1\}\sqrt{X^2+Y^2}\cos\left(\tan^{-1}\left(\frac{Y}{X}\right)+\phi-\frac{3\pi}{4}\right),\{X\ge 0\colon 1,X<0\colon -1\}k\sqrt{X^2+Y^2}\sin\left(\tan^{-1}\left(\frac{Y}{X}\right)+\phi-\frac{3\pi}{4}\right)+Z\sqrt{1-k^2}\right)$$

This is almost complete, except there is another problem when including the $\tan^{-1}(\frac{Y}{X})$ term, when both X and Y are 0. This hence causes a $\frac{0}{0}$ problem. Demos automatically handles $\frac{Y}{0}$ cases, as $\lim_{x\to\pm\infty}(\tan^{-1}(x))=\frac{\pi}{2}$, hence outputs $\tan^{-1}(\frac{Y}{0})=\frac{\pi}{2}$. However, the equation breaks down when both X and Y equal 0). This can be simply fixed however, by using the fact that the point $P(0{,}0{,}0)$ lies on the origin of the normal xy plane, hence all the terms except for the $Z\sqrt{2-2k^2}$ term become unnecessary. This gives:

$$P^*(0{,}0,Z)=\left(0,\frac{Z\sqrt{2-2k^2}}{z}\right)$$

$$P^*(0{,}0,Z)=\left(0,\frac{Z\sqrt{2-2k^2}}{z}\right)$$

Finally, these two cases can be combined as follows, to give the final projection function P:

$$P_1(X,Y,Z)=\frac{\sqrt{2}}{z}\left(\{X\ge 0\colon 1,X<0\colon -1\}\sqrt{X^2+Y^2}\cos\left(\tan^{-1}\left(\frac{Y}{X}\right)+\phi-\frac{3\pi}{4}\right),k\{X\ge 0\colon 1,X<0\colon -1\}\sqrt{X^2+Y^2}\sin\left(\tan^{-1}\left(\frac{Y}{X}\right)+\phi-\frac{3\pi}{4}\right)+Z\sqrt{1-k^2}\right)$$

$$P_2(X,Y,Z)=\left(0,\frac{Z\sqrt{2-2k^2}}{z}\right)$$

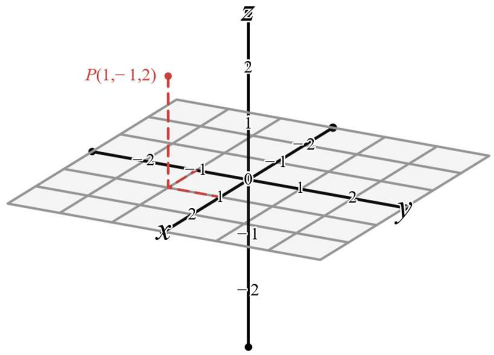
$$P(X,Y,Z)=\{|X|>0\colon P_1(X,Y,Z),X=0\colon\{|Y|>0\colon P_1(X,Y,Z),Y=0\colon P_2(X,Y,Z)\}\}$$

Hence, this was implemented using the following LaTeX code, and can also be found under the "Visualizing the 3D space" folder in the simulation:

```
P_{1}\left(X,Y,Z\right)=\frac{1}{z}\left(\left\{X\ge0:1,X<0:-1\right\}\sqrt{2Y^{2}+2X^{2}}\cos\left(\tan^{-1}\left(\frac{Y}{X}\right)+\phi-\frac{3\pi}{4}\right),\ k\left\{X\ge0:1,X<0:-1\right\}\sqrt{2Y^{2}+2X^{2}}\sin\left(\tan^{-1}\left(\frac{Y}{X}\right)+\phi-\frac{3\pi}{4}\right)+Z\sqrt{2-2k^{2}}\right)
P_{2}\left(X,Y,Z\right)=\left(0,\ \frac{Z\sqrt{2-2k^{2}}}{z}\right)
P\left(X,Y,Z\right)=\left\{\left|X\right|>0:P_{1}\left(X,Y,Z\right),X=0:\left\{\left|Y\right|>0:P_{1}\left(X,Y,Z\right),Y=0:P_{2}\left(X,Y,Z\right)\right\}\right\}
```

Implementing this projection function allows for points to be plotted on the new 3D space. An example of this is shown in Figure 08, with $P(1,-1,2)$.

Figure 08: Plotting $P(1,-1,2)$



| Section 6.2.1: Explaining the motions of three bodies interacting through gravity and deriving the equations required for the simulation

Consider three bodies M_1, M_2 and M_3, with positions $M_1(X_1, Y_1, Z_1)$, $M_2(X_2, Y_2, Z_2)$ and $M_3(X_3, Y_3, Z_3)$ respectively, where $X_1, Y_1, \ldots X_3, Y_3, Z_3$ are all functions of time t. Furthermore, let M_1, M_2 and M_3 have masses of m_1, m_2 and m_3 respectively. Let M_1 have an initial position of (x_1, y_1, z_1), let M_2 have an initial position of (x_2, y_2, z_2). Let M_3 have an initial position of (x_3, y_3, z_3). Let M_1 have an initial velocity vector V_1 with $V_1 = [v_{1x}, v_{1y}, v_{1z}]$. In the same fashion, let M_2 have velocity vector $V_2 = [v_{2x}, v_{2y}, v_{2z}]$ and let M_3 have velocity vector $V_3 = [v_{3x}, v_{3y}, v_{3z}]$. Finally, let G be the gravitational constant.

The motions of the three bodies can be derived using both Newtonian or Lagrangian mechanics. However, the use of Newtonian mechanics with $\vec{F} = m\vec{a}$ requires more steps in this scenario (such as resolving vectors), hence Lagrangian mechanics were used to derive the motions of the three bodies. The notation $\dot{f}$ to represent $\frac{d}{dt}f$, and $\ddot{f}$ to represent $\frac{d^2}{dt^2}f$ will be used.

Consider first solving for $X_1(t)$. The kinetic energy for M_1 can be found using the formula $K.E. = \frac{1}{2}mv^2$ [Khan Academy. (2022)] The total speed for M_1 is the sum of the individual speeds in the x, y and z directions, which are $\dot{X}_1$, $\dot{Y}_1$ and $\dot{Z}_1$ (As velocity is the derivative of position). This gives the total kinetic energy of M_1:

$$K.E. = \frac{1}{2}m_1\dot{X}_1^{\,2} + \frac{1}{2}m_1\dot{Y}_1^{\,2} + \frac{1}{2}m_1\dot{Z}_1^{\,2}$$

Then, the potential energy of M_1 can be found. The potential energy for one body in a system of two bodies, is given by $P.E. = -\frac{Gm_1m_2}{r}$ where m_1 and m_2 are the masses of the two bodies, and r is the distance between the two bodies [Khan Academy. (2018)]. Note that this distance can be found using the Pythagoras theorem, by taking the squares of the differences between the points, as shown below. Finally, for three bodies, the total potential energy is the sum of the potential energies in each two-body system. Below is hence the total potential energy for M_1:

$$P.E. = -\frac{Gm_1m_2}{\sqrt{(X_1-X_2)^2+(Y_1-Y_2)^2+(Z_1-Z_2)^2}} - \frac{Gm_1m_2}{\sqrt{(X_1-X_2)^2+(Y_1-Y_2)^2+(Z_1-Z_2)^2}}$$

We then have the Lagrangian:

$$L = K.E. - P.E.$$

This can now be substituted into the Euler-Lagrange equation as is typical with Lagrangian mechanics, which has the following form:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x_1}} = \frac{\partial L}{\partial x_1}$$

Differentiating and simplifying, we have:

$$\frac{d}{dt} m_1 \dot{X}_1 = -\frac{G m_1 m_2 (X_1 - X_2)}{((X_1 - X_2)^2 + (Y_1 - Y_2)^2 + (Z_1 - Z_2)^2)^{\frac{3}{2}}} - \frac{G m_1 m_3 (X_1 - X_3)}{((X_1 - X_3)^2 + (Y_1 - Y_3)^2 + (Z_1 - Z_3)^2)^{\frac{3}{2}}}$$

$$m_1 \ddot{X}_1 = -\frac{G m_1 m_2 (X_1 - X_2)}{((X_1 - X_2)^2 + (Y_1 - Y_2)^2 + (Z_1 - Z_2)^2)^{\frac{3}{2}}} - \frac{G m_1 m_3 (X_1 - X_3)}{((X_1 - X_3)^2 + (Y_1 - Y_3)^2 + (Z_1 - Z_3)^2)^{\frac{3}{2}}}$$

$$\ddot{X}_1 = \frac{G m_2 (X_2 - X_1)}{((X_1 - X_2)^2 + (Y_1 - Y_2)^2 + (Z_1 - Z_2)^2)^{\frac{3}{2}}} + \frac{G m_3 (X_3 - X_1)}{((X_1 - X_3)^2 + (Y_1 - Y_3)^2 + (Z_1 - Z_3)^2)^{\frac{3}{2}}}$$

Following the same process for the remaining 8 functions, which are the x, y and z co-ordinates for M_1, M_2 and M_3, we have the final differential equations which model the 3 bodies interacting through gravity:

$$\ddot{X}_1 = \frac{G m_2 (X_2 - X_1)}{((X_1 - X_2)^2 + (Y_1 - Y_2)^2 + (Z_1 - Z_2)^2)^{\frac{3}{2}}} + \frac{G m_3 (X_3 - X_1)}{((X_1 - X_3)^2 + (Y_1 - Y_3)^2 + (Z_1 - Z_3)^2)^{\frac{3}{2}}}$$

$$\ddot{Y}_1 = \frac{G m_2 (Y_2 - Y_1)}{((X_1 - X_2)^2 + (Y_1 - Y_2)^2 + (Z_1 - Z_2)^2)^{\frac{3}{2}}} + \frac{G m_3 (Y_3 - Y_1)}{((X_1 - X_3)^2 + (Y_1 - Y_3)^2 + (Z_1 - Z_3)^2)^{\frac{3}{2}}}$$

$$\ddot{Z}_1 = \frac{G m_2 (Z_2 - Z_1)}{((X_1 - X_2)^2 + (Y_1 - Y_2)^2 + (Z_1 - Z_2)^2)^{\frac{3}{2}}} + \frac{G m_3 (Z_3 - Z_1)}{((X_1 - X_3)^2 + (Y_1 - Y_3)^2 + (Z_1 - Z_3)^2)^{\frac{3}{2}}}$$

$$\ddot{X}_2 = \frac{G m_1 (X_1 - X_2)}{((X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2)^{\frac{3}{2}}} + \frac{G m_3 (X_3 - X_2)}{((X_2 - X_3)^2 + (Y_2 - Y_3)^2 + (Z_2 - Z_3)^2)^{\frac{3}{2}}}$$

$$\ddot{Y}_2 = \frac{G m_1 (Y_1 - Y_2)}{((X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2)^{\frac{3}{2}}} + \frac{G m_3 (Y_3 - Y_2)}{((X_2 - X_3)^2 + (Y_2 - Y_3)^2 + (Z_2 - Z_3)^2)^{\frac{3}{2}}}$$

$$\ddot{Z}_2 = \frac{G m_1 (Z_1 - Z_2)}{((X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2)^{\frac{3}{2}}} + \frac{G m_3 (Z_3 - Z_2)}{((X_2 - X_3)^2 + (Y_2 - Y_3)^2 + (Z_2 - Z_3)^2)^{\frac{3}{2}}}$$

$$\ddot{X}_3 = \frac{G m_1 (X_1 - X_3)}{((X_3 - X_1)^2 + (Y_3 - Y_1)^2 + (Z_3 - Z_1)^2)^{\frac{3}{2}}} + \frac{G m_2 (X_2 - X_3)}{((X_3 - X_2)^2 + (Y_3 - Y_2)^2 + (Z_3 - Z_2)^2)^{\frac{3}{2}}}$$

$$\ddot{Y}_3 = \frac{G m_1 (Y_1 - Y_3)}{((X_3 - X_1)^2 + (Y_3 - Y_1)^2 + (Z_3 - Z_1)^2)^{\frac{3}{2}}} + \frac{G m_2 (Y_2 - Y_3)}{((X_3 - X_2)^2 + (Y_3 - Y_2)^2 + (Z_3 - Z_2)^2)^{\frac{3}{2}}}$$

$$\ddot{Z}_3 = \frac{G m_1 (Z_1 - Z_3)}{((X_3 - X_1)^2 + (Y_3 - Y_1)^2 + (Z_3 - Z_1)^2)^{\frac{3}{2}}} + \frac{G m_2 (Z_2 - Z_3)}{((X_3 - X_2)^2 + (Y_3 - Y_2)^2 + (Z_3 - Z_2)^2)^{\frac{3}{2}}}$$

| Section 6.2.2: Approximating Solutions to the Differential Equations

The equations derived in the previous section have no analytic solution, meaning that $X_1, Y_1, \dots X_3, Y_3, Z_3$ cannot be written in terms of finite mathematical expressions. However, the solutions can be approximated. Consider the following definition of the first derivative of a function f of time t:

$$\frac{df}{dt} = \lim_{\Delta \to 0} \frac{f(t+\Delta) - f(t)}{\Delta}$$

Now, substituting this definition into itself, results in the definition of the second derivative:

$$\frac{d^2 f}{dt^2} = \lim_{\Delta \to 0} \frac{\frac{f(t+2\Delta) - f(t+\Delta)}{\Delta} - \frac{f(t+\Delta) - f(t)}{\Delta}}{\Delta}$$

Simplifying, we have:

$$\frac{d^2 f}{dt^2} = \lim_{\Delta \to 0} \frac{f(t+2\Delta) - 2f(t+\Delta) + f(t)}{\Delta^2}$$

As Δ approaches 0, we also have that the derivative at t approaches being the same as the derivative at $t - \Delta$. Hence substituting $t - \Delta$ for t, we also have the following definition of the second derivative:

$$\frac{d^2 f}{dt^2} = \lim_{\Delta \to 0} \frac{f(t+\Delta) - 2f(t) + f(t-\Delta)}{\Delta^2}$$

Now, using $X_1(t)$ as an example, we have the equation derived before:

$$\ddot{X}_1 = \frac{Gm_2(X_2 - X_1)}{((X_1 - X_2)^2 + (Y_1 - Y_2)^2 + (Z_1 - Z_2)^2)^{\frac{3}{2}}} + \frac{Gm_3(X_3 - X_1)}{((X_1 - X_3)^2 + (Y_1 - Y_3)^2 + (Z_1 - Z_3)^2)^{\frac{3}{2}}}$$

Hence, for sufficiently small Δ the following approximation holds:

$$\frac{X_1(t+\Delta) - 2X_1(t) + X_1(t-\Delta)}{\Delta^2} \approx \frac{Gm_2(X_2 - X_1)}{((X_1 - X_2)^2 + (Y_1 - Y_2)^2 + (Z_1 - Z_2)^2)^{\frac{3}{2}}} + \frac{Gm_3(X_3 - X_1)}{((X_1 - X_3)^2 + (Y_1 - Y_3)^2 + (Z_1 - Z_3)^2)^{\frac{3}{2}}}$$

Hence, if $X_1(t)$ and $X_1(t-\Delta)$ is known, $X_1(t + \Delta t)$ can be found. This can be rearranged to give the following recursive formula which can ultimately used to approximate solutions to the differential equations:

$$X_1(t+\Delta) \approx \frac{Gm_2(X_2 - X_1)\Delta^2}{((X_1 - X_2)^2 + (Y_1 - Y_2)^2 + (Z_1 - Z_2)^2)^{\frac{3}{2}}} + \frac{Gm_3(X_3 - X_1)\Delta^2}{((X_1 - X_3)^2 + (Y_1 - Y_3)^2 + (Z_1 - Z_3)^2)^{\frac{3}{2}}} + 2X_1(t) - X_1(t-\Delta)$$

Hence, to solve the differential equations, $X_1, Y_1, \dots X_3, Y_3, Z_3$ were written in Desmos as lists of two numbers, one for the respective function calculated at $t - \Delta$ and the other for the respective function calculated at t. For example, $X_1(t)$ was written in the form shown below:

$$X_1 = [X_1(t-\Delta), X_1(t)]$$

An action was defined in Desmos that, every millisecond, would use the recursive formula found previously to calculate $X_1(t+\Delta)$, with a very small value of Δ. Then, the list would be updated to the following:

$$X_1 = [X_1(t), X_1(t+\Delta)]$$

This means that $X_1[2]$ (which is the notation used to indicate the second number in the list X_1) would correspond to $X_1(t)$ at any time t. This was how the Desmos simulation was ultimately run, with an action A_1 being designed to perform the following recursive action every 1 millisecond:

$$A_1 = X_1 \rightarrow \left[X_1[2], \frac{Gm_2(X_2[2]-X_1[2])\Delta^2}{((X_1[2]-X_2[2])^2+(Y_1[2]-Y_2[2])^2+(Z_1[2]-Z_2[2])^2)^{\frac{3}{2}}} \right.$$
$$\left. + \frac{Gm_3(X_3[2]-X_1[2])\Delta^2}{((X_1[2]-X_3[2])^2+(Y_1[2]-Y_3[2])^2+(Z_1[2]-Z_3[2])^2)^{\frac{3}{2}}} + 2X_1[2] - X_1[1] \right]$$

As can be seen, performing this action updates the X_1 from $X_1 = [X_1(t), X_1(t+\Delta)]$ to $X_1 = [X_1(t+\Delta), X_1(t+2\Delta)]$. A similar process was used to define the remaining 8 functions $Y_1, Z_1, X_2 \ldots X_3, Y_3, Z_3$ in terms of recursive formulae and lists, and then actions $A_2, A_3, A_4 \ldots A_8, A_9$ were created in the same fashion for these functions. These recursive actions are shown below:

$$A_2 = Y_1 \rightarrow \left[Y_1[2], \frac{Gm_2(Y_2[2]-Y_1[2])}{((X_1[2]-X_2[2])^2+(Y_1[2]-Y_2[2])^2+(Z_1[2]-Z_2[2])^2)^{\frac{3}{2}}} \right.$$
$$\left. + \frac{Gm_3(Y_3[2]-Y_1[2])}{((X_1[2]-X_3[2])^2+(Y_1[2]-Y_3[2])^2+(Z_1[2]-Z_3[2])^2)^{\frac{3}{2}}} + 2Y_1[2] - Y_1[1] \right]$$

$$A_3 = Z_1 \rightarrow \left[Z_1[2], \frac{Gm_2(Z_2[2]-Z_1[2])}{((X_1[2]-X_2[2])^2+(Y_1[2]-Y_2[2])^2+(Z_1[2]-Z_2[2])^2)^{\frac{3}{2}}} \right.$$
$$\left. + \frac{Gm_3(Z_3[2]-Z_1[2])}{((X_1[2]-X_3[2])^2+(Y_1[2]-Y_3[2])^2+(Z_1[2]-Z_3[2])^2)^{\frac{3}{2}}} + 2Z_1[2] - Z_1[1] \right]$$

$$A_4 = X_2 \rightarrow \left[X_2[2], \frac{Gm_1(X_1[2]-X_2[2])}{((X_2[2]-X_1[2])^2+(Y_2[2]-Y_1[2])^2+(Z_2[2]-Z_1[2])^2)^{\frac{3}{2}}} \right.$$
$$\left. + \frac{Gm_3(X_3[2]-X_2[2])}{((X_2[2]-X_3[2])^2+(Y_2[2]-Y_3[2])^2+(Z_2[2]-Z_3[2])^2)^{\frac{3}{2}}} + 2X_2[2] - X_2[1] \right]$$

$$A_5 = Y_2 \rightarrow \left[Y_2[2], \frac{Gm_1(Y_1[2]-Y_2[2])}{((X_2[2]-X_1[2])^2+(Y_2[2]-Y_1[2])^2+(Z_2[2]-Z_1[2])^2)^{\frac{3}{2}}} \right.$$
$$\left. + \frac{Gm_3(Y_3[2]-Y_2[2])}{((X_2[2]-X_3[2])^2+(Y_2[2]-Y_3[2])^2+(Z_2[2]-Z_3[2])^2)^{\frac{3}{2}}} + 2Y_2[2] - Y_2[1] \right]$$

$$A_6 = Z_2 \rightarrow \left[Z_2[2], \frac{Gm_1(Z_1[2]-Z_2[2])}{((X_2[2]-X_1[2])^2+(Y_2[2]-Y_1[2])^2+(Z_2[2]-Z_1[2])^2)^{\frac{3}{2}}} \right.$$
$$\left. + \frac{Gm_3(Z_3[2]-Z_2[2])}{((X_2[2]-X_3[2])^2+(Y_2[2]-Y_3[2])^2+(Z_2[2]-Z_3[2])^2)^{\frac{3}{2}}} + 2Z_2[2] - Z_2[1] \right]$$

$$A_7 = X_3 \to \left[X_3[2], \frac{Gm_1(X_1[2]-X_3[2])}{((X_3[2]-X_1[2])^2+(Y_3[2]-Y_1[2])^2+(Z_3[2]-Z_1[2])^2)^{\frac{3}{2}}} + \frac{Gm_2(X_2[2]-X_3[2])}{((X_3[2]-X_2[2])^2+(Y_3[2]-Y_2[2])^2+(Z_3[2]-Z_2[2])^2)^{\frac{3}{2}}} + 2X_3[2] - X_3[1]\right]$$

$$A_8 = Y_3 \to \left[Y_3[2], \frac{Gm_1(Y_1[2]-Y_3[2])}{((X_3[2]-X_1[2])^2+(Y_3[2]-Y_1[2])^2+(Z_3[2]-Z_1[2])^2)^{\frac{3}{2}}} + \frac{Gm_2(Y_2[2]-Y_3[2])}{((X_3[2]-X_2[2])^2+(Y_3[2]-Y_2[2])^2+(Z_3[2]-Z_2[2])^2)^{\frac{3}{2}}} + 2Y_3[2] - Y_3[1]\right]$$

$$A_9 = Z_3 \to \left[Z_3[2], \frac{Gm_1(Z_1[2]-Z_3[2])}{((X_3[2]-X_1[2])^2+(Y_3[2]-Y_1[2])^2+(Z_3[2]-Z_1[2])^2)^{\frac{3}{2}}} + \frac{Gm_2(Z_2[2]-Z_3[2])}{((X_3[2]-X_2[2])^2+(Y_3[2]-Y_2[2])^2+(Z_3[2]-Z_2[2])^2)^{\frac{3}{2}}} + 2Z_3[2] - Z_3[1]\right]$$

Hence, this means that the positions of the three bodies were written in Desmos as $M_1(X_1[2], Y_1[2], Z_1[2])$, $M_2(X_2[2], Y_2[2], Z_2[2])$, and $M_3(X_3[2], Y_3[2], Z_3[2])$, as it is the second element in each of the lists which correspond to the value of each of the functions at any given time t.

Similarly, to update the velocity vectors V_1, V_2 and V_3, the following actions were also defined. As can be seen, these actions also simply make use of the $v = \frac{d}{t}$ formula:

$$A_{10} = V_1 \to \left[\frac{X_1[2]-X_1[1]}{\Delta}, \frac{Y_1[2]-Y_1[1]}{\Delta}, \frac{Z_1[2]-Z_1[1]}{\Delta}\right]$$

$$A_{11} = V_2 \to \left[\frac{X_2[2]-X_2[1]}{\Delta}, \frac{Y_2[2]-Y_2[1]}{\Delta}, \frac{Z_2[2]-Z_2[1]}{\Delta}\right]$$

$$A_{12} = V_3 \to \left[\frac{X_3[2]-X_3[1]}{\Delta}, \frac{Y_3[2]-Y_3[1]}{\Delta}, \frac{Z_3[2]-Z_3[1]}{\Delta}\right]$$

Then, the action U was set such that $U = A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_{10}, A_{11}, A_{12}$, meaning when U is run each millisecond using the Desmos "ticker" function, the positions and velocities for the 3 bodies are updated.

However, note that there is one case in which the recursive formulae found do not work. As finding the value of each of $X_1, Y_1, ... X_3, Y_3, Z_3$ at $t + \Delta$ requires knowing the value of the functions at t and at $t - \Delta$, when $t = 0$, (or in the initial conditions) there is no known value of each of the functions at $t - \Delta$. However, there is a quick method to avoid this, which allows for the initial velocities of the 3 bodies to also be accounted for. Consider X_1 as an example. Note that, if the original speed of M_1 in the x direction was previously said to be v_{1x}, then using the simple formula for speed of $v = \frac{d}{t}$, we have:

$$v_x = \frac{X_1(0) - X_1(0-\Delta)}{\Delta}$$
$$X_1(0-\Delta) = X_1(0) - \Delta v_x$$

Given that it was said before that the initial position for M_1 is (x_1, y_1, z_1), we have:

$$X_1(0-\Delta) = x_1 - \Delta v_x$$

This further means that the initial conditions for the list X_1 is:

$$X_1 = [x_1 - \Delta v_{1x}, x_1]$$

In a similar way, the initial conditions for list Y_1 is $[y_1 - \Delta v_{1y}, y_1]$, and so on for $Z_1, X_2 \ldots X_3, Y_3, Z_3$. Hence, when the "Reset" button is pressed, X_1 is reset to $[x_1 - \Delta v_{1x}, x_1]$, Y_1 is set to $[y_1 - \Delta v_{1y}, y_1]$, and so on for the remaining 7 functions for the positions of the 3 bodies.

Hence, to summarise, the numerical solutions to the differential equations are found in the following three steps:

1. The positions of M_1, M_2 and M_3 are stored as $M_1(X_1[2], Y_1[2], Z_1[2])$, $M_2(X_2[2], Y_2[2], Z_2[2])$, and $M_3(X_3[2], Y_3[2], Z_3[2])$, and with initial positions $M_1(x_1, y_1, z_1)$, $M_2(x_2, y_2, z_2)$ and $M_3(x_3, y_3, z_3)$ respectively. The velocity vectors for M_1, M_2 and M_3 are stored as V_1, V_2 and V_3, with initial conditions $V_1 = [v_{1x}, v_{1y}, v_{1z}]$, $V_2 = [v_{2x}, v_{2y}, v_{2z}]$ and $V_3 = [v_{3x}, v_{3y}, v_{3z}]$.
2. Recursive formulae derived from approximations for the second derivative are used to calculate the values of each of these functions at $t + \Delta$ using the known values at t and $t - \Delta$. This is done using an action U which runs every millisecond after the "Start/Stop" button is pressed.
3. When the "Reset" button is pressed, the initial conditions are reset.

Hence, this was implemented using the following LaTeX code, and can also be found under the "Position and Velocity Updates" folder in the simulation:

X_{1}=\left[_,_\right]
Y_{1}=\left[_,_\right]
Z_{1}=\left[_,_\right]
X_{2}=\left[_,_\right]
Y_{2}=\left[_,_\right]
Z_{2}=\left[_,_\right]
X_{3}=\left[_,_\right]
Y_{3}=\left[_,_\right]
Z_{3}=\left[_,_\right]
M_{1}=P\left(X_{1}\left[2\right],Y_{1}\left[2\right],Z_{1}\left[2\right]\right)
M_{2}=P\left(X_{2}\left[2\right],Y_{2}\left[2\right],Z_{2}\left[2\right]\right)
M_{3}=P\left(X_{3}\left[2\right],Y_{3}\left[2\right],Z_{3}\left[2\right]\right)
V_{1}=\left[_,_,_\right]
V_{2}=\left[_,_,_\right]
V_{3}=\left[_,_,_\right]
A_{1}=X_{1}\to\left[X_{1}\left[2\right],\frac{Gm_{2}\left(X_{2}\left[2\right]-X_{1}\left[2\right]\right)\Delta^{2}}{\left(\left(X_{1}\left[2\right]-X_{2}\left[2\right]\right)^{2}+\left(Y_{1}\left[2\right]-Y_{2}\left[2\right]\right)^{2}+\left(Z_{1}\left[2\right]-Z_{2}\left[2\right]\right)^{2}\right)^{\frac{3}{2}}}+\frac{Gm_{3}\left(X_{3}\left[2\right]-X_{1}\left[2\right]\right)\Delta^{2}}{\left(\left(X_{1}\left[2\right]-X_{3}\left[2\right]\right)^{2}+\left(Y_{1}\left[2\right]-Y_{3}\left[2\right]\right)^{2}+\left(Z_{1}\left[2\right]-Z_{3}\left[2\right]\right)^{2}\right)^{\frac{3}{2}}}+2X_{1}\left[2\right]-X_{1}\left[1\right]\right]

A_{2}=Y_{1}\to\left[Y_{1}\left[2\right],\frac{Gm_{2}\left(Y_{2}\left[2\right]-Y_{1}\left[2\right]\right)\Delta^{2}}{\left(\left(X_{1}\left[2\right]-X_{2}\left[2\right]\right)^{2}+\left(Y_{1}\left[2\right]-Y_{2}\left[2\right]\right)^{2}+\left(Z_{1}\left[2\right]-Z_{2}\left[2\right]\right)^{2}\right)^{\frac{3}{2}}}+\frac{Gm_{3}\left(Y_{3}\left[2\right]-Y_{1}\left[2\right]\right)\Delta^{2}}{\left(\left(X_{1}\left[2\right]-X_{3}\left[2\right]\right)^{2}+\left(Y_{1}\left[2\right]-Y_{3}\left[2\right]\right)^{2}+\left(Z_{1}\left[2\right]-Z_{3}\left[2\right]\right)^{2}\right)^{\frac{3}{2}}}+2Y_{1}\left[2\right]-Y_{1}\left[1\right]\right]
A_{3}=Z_{1}\to\left[Z_{1}\left[2\right],\frac{Gm_{2}\left(Z_{2}\left[2\right]-Z_{1}\left[2\right]\right)\Delta^{2}}{\left(\left(X_{1}\left[2\right]-X_{2}\left[2\right]\right)^{2}+\left(Y_{1}\left[2\right]-Y_{2}\left[2\right]\right)^{2}+\left(Z_{1}\left[2\right]-Z_{2}\left[2\right]\right)^{2}\right)^{\frac{3}{2}}}+\frac{Gm_{3}\left(Z_{3}\left[2\right]-Z_{1}\left[2\right]\right)\Delta^{2}}{\left(\left(X_{1}\left[2\right]-X_{3}\left[2\right]\right)^{2}+\left(Y_{1}\left[2\right]-Y_{3}\left[2\right]\right)^{2}+\left(Z_{1}\left[2\right]-Z_{3}\left[2\right]\right)^{2}\right)^{\frac{3}{2}}}+2Z_{1}\left[2\right]-Z_{1}\left[1\right]\right]
A_{4}=X_{2}\to\left[X_{2}\left[2\right],\frac{Gm_{1}\left(X_{1}\left[2\right]-X_{2}\left[2\right]\right)\Delta^{2}}{\left(\left(X_{2}\left[2\right]-X_{1}\left[2\right]\right)^{2}+\left(Y_{2}\left[2\right]-Y_{1}\left[2\right]\right)^{2}+\left(Z_{2}\left[2\right]-Z_{1}\left[2\right]\right)^{2}\right)^{\frac{3}{2}}}+\frac{Gm_{3}\left(X_{3}\left[2\right]-X_{2}\left[2\right]\right)\Delta^{2}}{\left(\left(X_{2}\left[2\right]-X_{3}\left[2\right]\right)^{2}+\left(Y_{2}\left[2\right]-Y_{3}\left[2\right]\right)^{2}+\left(Z_{2}\left[2\right]-Z_{3}\left[2\right]\right)^{2}\right)^{\frac{3}{2}}}+2X_{2}\left[2\right]-X_{2}\left[1\right]\right]
A_{5}=Y_{2}\to\left[Y_{2}\left[2\right],\frac{Gm_{1}\left(Y_{1}\left[2\right]-Y_{2}\left[2\right]\right)\Delta^{2}}{\left(\left(X_{2}\left[2\right]-X_{1}\left[2\right]\right)^{2}+\left(Y_{2}\left[2\right]-Y_{1}\left[2\right]\right)^{2}+\left(Z_{2}\left[2\right]-Z_{1}\left[2\right]\right)^{2}\right)^{\frac{3}{2}}}+\frac{Gm_{3}\left(Y_{3}\left[2\right]-Y_{2}\left[2\right]\right)\Delta^{2}}{\left(\left(X_{2}\left[2\right]-X_{3}\left[2\right]\right)^{2}+\left(Y_{2}\left[2\right]-Y_{3}\left[2\right]\right)^{2}+\left(Z_{2}\left[2\right]-Z_{3}\left[2\right]\right)^{2}\right)^{\frac{3}{2}}}+2Y_{2}\left[2\right]-Y_{2}\left[1\right]\right]
A_{6}=Z_{2}\to\left[Z_{2}\left[2\right],\frac{Gm_{1}\left(Z_{1}\left[2\right]-Z_{2}\left[2\right]\right)\Delta^{2}}{\left(\left(X_{2}\left[2\right]-X_{1}\left[2\right]\right)^{2}+\left(Y_{2}\left[2\right]-Y_{1}\left[2\right]\right)^{2}+\left(Z_{2}\left[2\right]-Z_{1}\left[2\right]\right)^{2}\right)^{\frac{3}{2}}}+\frac{Gm_{3}\left(Z_{3}\left[2\right]-Z_{2}\left[2\right]\right)\Delta^{2}}{\left(\left(X_{2}\left[2\right]-X_{3}\left[2\right]\right)^{2}+\left(Y_{2}\left[2\right]-

Y_{3}\left[2\right]\right)^{2}+\left(Z_{2}\left[2\right]-Z_{3}\left[2\right]\right)^{2}\right)^{\frac{3}{2}}}+2Z_{2}\left[2\right]-Z_{2}\left[1\right]\right]
A_{7}=X_{3}\to\left[X_{3}\left[2\right],\frac{Gm_{1}\left(X_{1}\left[2\right]-X_{3}\left[2\right]\right)\Delta^{2}}{\left(\left(X_{3}\left[2\right]-X_{1}\left[2\right]\right)^{2}+\left(Y_{3}\left[2\right]-Y_{1}\left[2\right]\right)^{2}+\left(Z_{3}\left[2\right]-Z_{1}\left[2\right]\right)^{2}\right)^{\frac{3}{2}}}+\frac{Gm_{2}\left(X_{2}\left[2\right]-X_{3}\left[2\right]\right)\Delta^{2}}{\left(\left(X_{3}\left[2\right]-X_{2}\left[2\right]\right)^{2}+\left(Y_{3}\left[2\right]-Y_{2}\left[2\right]\right)^{2}+\left(Z_{3}\left[2\right]-Z_{2}\left[2\right]\right)^{2}\right)^{\frac{3}{2}}}+2X_{3}\left[2\right]-X_{3}\left[1\right]\right]
A_{8}=Y_{3}\to\left[Y_{3}\left[2\right],\frac{Gm_{1}\left(Y_{1}\left[2\right]-Y_{3}\left[2\right]\right)\Delta^{2}}{\left(\left(X_{3}\left[2\right]-X_{1}\left[2\right]\right)^{2}+\left(Y_{3}\left[2\right]-Y_{1}\left[2\right]\right)^{2}+\left(Z_{3}\left[2\right]-Z_{1}\left[2\right]\right)^{2}\right)^{\frac{3}{2}}}+\frac{Gm_{2}\left(Y_{2}\left[2\right]-Y_{3}\left[2\right]\right)\Delta^{2}}{\left(\left(X_{3}\left[2\right]-X_{2}\left[2\right]\right)^{2}+\left(Y_{3}\left[2\right]-Y_{2}\left[2\right]\right)^{2}+\left(Z_{3}\left[2\right]-Z_{2}\left[2\right]\right)^{2}\right)^{\frac{3}{2}}}+2Y_{3}\left[2\right]-Y_{3}\left[1\right]\right]
A_{9}=Z_{3}\to\left[Z_{3}\left[2\right],\frac{Gm_{1}\left(Z_{1}\left[2\right]-Z_{3}\left[2\right]\right)\Delta^{2}}{\left(\left(X_{3}\left[2\right]-X_{1}\left[2\right]\right)^{2}+\left(Y_{3}\left[2\right]-Y_{1}\left[2\right]\right)^{2}+\left(Z_{3}\left[2\right]-Z_{1}\left[2\right]\right)^{2}\right)^{\frac{3}{2}}}+\frac{Gm_{2}\left(Z_{2}\left[2\right]-Z_{3}\left[2\right]\right)\Delta^{2}}{\left(\left(X_{3}\left[2\right]-X_{2}\left[2\right]\right)^{2}+\left(Y_{3}\left[2\right]-Y_{2}\left[2\right]\right)^{2}+\left(Z_{3}\left[2\right]-Z_{2}\left[2\right]\right)^{2}\right)^{\frac{3}{2}}}+2Z_{3}\left[2\right]-Z_{3}\left[1\right]\right]
A_{10}=V_{1}\to\left[\frac{X_{1}\left[2\right]-X_{1}\left[1\right]}{\Delta},\frac{Y_{1}\left[2\right]-Y_{1}\left[1\right]}{\Delta},\frac{Z_{1}\left[2\right]-Z_{1}\left[1\right]}{\Delta}\right]
A_{11}=V_{2}\to\left[\frac{X_{2}\left[2\right]-X_{2}\left[1\right]}{\Delta},\frac{Y_{2}\left[2\right]-Y_{2}\left[1\right]}{\Delta},\frac{Z_{2}\left[2\right]-Z_{2}\left[1\right]}{\Delta}\right]
A_{12}=V_{3}\to\left[\frac{X_{3}\left[2\right]-X_{3}\left[1\right]}{\Delta},\frac{Y_{3}\left[2\right]-Y_{3}\left[1\right]}{\Delta},\frac{Z_{3}\left[2\right]-Z_{3}\left[1\right]}{\Delta}\right]
U=A_{1},A_{2},A_{3},A_{4},A_{5},A_{6},A_{7},A_{8},A_{9},A_{10},A_{11},A_{12}

This was further implemented using the following LaTeX code, and can also be found under the "Velocity Vectors" folder in the simulation:

\left[M_{1},P\left(X_{1}\left[2\right]+V_{1}\left[1\right],Y_{1}\left[2\right]+V_{1}\left[2\right],Z_{1}\left[2\right]+V_{1}\left[3\right]\right)\right]

\left[M_{2},P\left(X_{2}\left[2\right]+V_{2}\left[1\right],Y_{2}\left[2\right]+V_{2}\left[2\right],Z_{2}\left[2\right]+V_{2}\left[3\right]\right)\right]
\left[M_{3},P\left(X_{3}\left[2\right]+V_{3}\left[1\right],Y_{3}\left[2\right]+V_{3}\left[2\right],Z_{3}\left[2\right]+V_{3}\left[3\right]\right)\right]

This was further implemented using the following LaTeX code, and can also be found under the "User Interface" folder in the simulation:

\left(\frac{35}{24},0.6\right)
R_{eset}=X_{1}\to\left[x_{1}-\Delta v_{1x},x_{1}\right],Y_{1}\to\left[y_{1}-\Delta v_{1y},y_{1}\right],Z_{1}\to\left[z_{1}-\Delta v_{1z},z_{1}\right],X_{2}\to\left[x_{2}-\Delta v_{2x},x_{2}\right],Y_{2}\to\left[y_{2}-\Delta v_{2y},y_{2}\right],Z_{2}\to\left[z_{2}-\Delta v_{2z},z_{2}\right],X_{3}\to\left[x_{3}-\Delta v_{3x},x_{3}\right],Y_{3}\to\left[y_{3}-\Delta v_{3y},y_{3}\right],Z_{3}\to\left[z_{3}-\Delta v_{3z},z_{3}\right],V_{1}\to\left[v_{1x},v_{1y},v_{1z}\right],V_{2}\to\left[v_{2x},v_{2y},v_{2z}\right],V_{3}\to\left[v_{3x},v_{3y},v_{3z}\right]
\left(\frac{43}{24},0.6\right)
P_{lay}=S\to1-S

| Section 6.3.1: User interface

Finally, the following aspects that are implemented using the following LaTeX code, and can also be found under the "User Interface" folder in the simulation, are simply used for aesthetic and user-friendliness purposes (such as the point used to rotate the simulation):

\left(2-\frac{\phi}{2\pi},\frac{k}{2}\right)
\operatorname{polygon}\left(\left(10,10\right),\left(-10,10\right),\left(-10,-10\right),\left(10,-10\right)\right)
\left[\left(1.125,0\right),\left(2.125,0\right)\right]
\operatorname{polygon}\left(\left(1.125,0.5\right),\left(2.125,0.5\right),\left(2.125,-0.5\right),\left(1.125,-0.5\right)\right)

| Section 7: Acknowledgements

I would primarily like to acknowledge several YouTube videos that really made the topic of differential equations clear to me. Especially the following source; myphysicsnotebook (2022). Three body problem simulation on Desmos. [online] YouTube. Available at: https://www.youtube.com/watch?v=lXzfm55e6e8 which discusses a simulation similar to mine, but in 2 dimensions. This is the video which gave me the idea of using lists to implement recursive formulae for the positions three bodies, but it must be noted that the final methodology and implementation is different in my simulation. Similarly, I would like to acknowledge Desmos online graphing calculator, for making this project possible.

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